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ON INVARIANT PROBABILITIES FOR RANDOM ROTATIONS

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To the memory of Shlomo Horowitz

ABSTRACT

It is shown that a spacially dependent random rotation process on the circle always possesses an invariant probability equivalent to Lebesgue measure.

0. The ergodic properties of an irrational rotation α of the circle Γ have been often studied from various points of view. The fact that Lebesque measure is the only invariant probability is elementary and does not depend on deep number theoretic properties of the particular irrational α but just on the fact that α is not a root of unity.

If we select several rotations $\{\alpha_1, \cdots, \alpha_n\}$ and toss an n-sided die at each instance of discrete time to determine which of the n rotation numbers will govern the next jump of a particle moving on the circle we again have an often studied object, for this procedure generates a rather simple random walk on the circle.

We obtain from the random walk a Markov operator T (acting on say the continuous functions) given by

$$(*) \qquad Tf(x) = \sum p_i f(x + \alpha_i)$$

where we have treated the circle as an additive group. Again Lebesgue measure is invariant and conditions for uniqueness of invariant probabilities are well known and understood. The operator given by (*) is expressed a convex combination of other Markov operators R_i where R_i is the rotation operator determined by α_i. We generalize once more by allowing the coefficients $\{p_i\}$ in

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equation (*) to be spacially dependent. For this non-spacially homogeneous random walk little is known. The natural questions are uniqueness of invariant probabilities and existence of an invariant probability weaker than Lebesque measure. (Lebesgue measure itself is not, in general, invariant for such a random rotation operator.) We prove existence of a Lebesgue equivalent invariant measure in the next section and discuss the uniqueness problem in the section following that.

1. This section is devoted to the existence of Haar equivalent invariant probabilities for certain spacially dependent random rotations of compact abelian groups. We use the L_1 contraction theory of Markov operators as initiated by Hopf and developed in Foguel's notes [3]. This will be the main reference for terminology, notation, and results which will be used in this section. Thus a (Hopf) Markov operator is a positive $L_1(m)$ contraction. The standard way of going from a Feller type operator as given by (*) to a Hopf type is to find a nonsingular reference measure m so that $T*m \ll m$. Then, via Radon–Nikodym derivatives, an operator is defined on $L_1(m)$. This operator we will denote by U and, as is customary, we will write U to the right of the functions (densities with respect to m) on which it acts.

The major tool for the result is Brunel's recent elegant characterization of operators U which have an invariant probability measure equivalent to m [2]. To this end, for each sequence $c = \{c_i\}$ of non-negative terms which sum to 1 we define $V = \Sigma c_i U^i$. The set of all such operators is called the class of derived operators (corresponding to U). The result of Brunel is this. The operator U has an invariant probability equivalent to m (that is, f in $L_1(m)$ so that $f > 0$ and $fU = f$) if and only if each derived operator is conservative in the sense of Hopf. An exposition of Brunel's theorem can also be found in Foguel's British Columbia mimeographed notes [4].

THEOREM. *Let $T = \Sigma p_i R_i$ be a (Feller) Markov operator on the circle G where $\{R_i : 1 \leqq i \leqq n\}$ are rotations of G and $\{p_i\}$ are strictly positive continuous functions on G. If at least one of the rotations is a Kronecker irrational then there exists an invariant probability which is equivalent to Haar measure.*

REMARK. For random rotations of a general compact group the presence of one rotation, all of whose powers generate dense orbits, will give the corresponding theorem.

PROOF. Let $d\sigma = fdh$ be a Haar continuous finite measure on G. Then

$$(g, T * \sigma) = (Tg, \sigma) = \left(\sum p_i R_i g, fdh\right) = \left(g, \sum R_i * p_i fdh\right)$$

showing that $T * \sigma$ is Haar continuous so that h is a non-singular reference measure. The induced Hopf Markov operator on $L_1(h)$ will be denoted by U. Let us suppose that V is a derived Markov operator of U (on $L_1(h)$ again) which is not Hopf conservative. Then $hC_v < 1$ and the function g which is the indicator of C_v satisfies $Vg \geqq g$. This implies $U^i g \geqq g$ for each i in the support of $\{c\}$. Thus T^i has a set C_v which is Haar essentially invariant and has Haar measure less than 1. By the form of the operator T this can happen only if $hC_v = 0$. So by convolution arguments we obtain that V is a Hopf dissipative operator.

Now a totally non-conservative operator has some redeeming features. The property we need is that such an operator has a σ-finite subinvariant measure $d\lambda = fdh$. (See [3], p. 17]). We can, in fact, take $f = \Sigma 1v^n$ so $d\lambda$ is equivalent to dh. Now λ is also subinvariant for $V*$ for

$$\int (1_A V) d\lambda \leqq \int 1_A d\lambda$$

by [3, p. 76] which means

$$\int 1_A (Sf) dh \leqq \int 1_A fdm$$

where S is the $\{c_i\}$ convex power series in T. Thus $Sf \leqq f$ almost everywhere with respect to Haar measure. Recall that f was defined by $\Sigma 1 V^n$ and observe that U (and also each power U^i) takes continuous functions into functions with continuous versions. It follows that f is lower semicontinuous on G. Now the essential minimizing set of f is closed and nonempty. Since f and Sf have lower semicontinuous versions this set is invariant under S. Again the form of T implies no closed invariant nontrivial sets for even S. This forces f to be h-essentially constant and V to be conservative contrary to assumption. This contradiction allows us to conclude the proof with an application of Brunel's theorem.

2. The investigation of random rotations on the circle was initiated in hopes of gaining understanding of the similar problem of random translations on the line. In [5] it was shown that for two noncommensurable translations on **R** with strictly positive Borel coefficient functions there is at most one invariant probability. Moreover when there is an invariant probability it must be Lebesgue equivalent. This result generalized a theorem of Norman and Pickands to be found in Norman's book on learning theory [7] and in slightly strengthened form in Norman's SIAM review [8]. Frank Norman has informed the author that a third term for the identity translation may always be allowed in addition with the same conclusion holding so that the analysis of [5] can be applied to a learning model considered by Kac [6].

The covering argument of [5] for the line can be used here for the circle to obtain the following.

THEOREM. *Let p be continuous on Γ and satisfy $0 < p < 1$. Let α be a unimodular nonroot of unity and let β be a unimodular root of unity. Define T on $C(\Gamma)$ by*

$$Tf(x) = p(x)f(x+\alpha) + q(x)f(x+\alpha+\beta).$$

Then T is uniquely ergodic and the invariant probability is absolutely continuous with respect to Lebesgue measure.

The technique of the proof is to use a covering map of the circle by the circle to induce a deterministic irrational rotation on the covered circle. Lebesgue continuity is then established by shuffling null sets back and forth. The invariant probability is not explicitly identified. Again we leave to the interested reader the task of adopting the argument of [5] for the details.

A special case of [1] can be given a Fourier proof for the circle.

THEOREM. *Let p be continuous on the circle Γ and satisfy $0 \leqq p < 1$. Let α be a unimodular number which is not a root of unity. Define T on $C(\Gamma)$ by*

$$Tf(x) = p(x)f(x) + q(x)f(x+\alpha)$$

where $q(x) = 1 - p(x)$. Then T has exactly one invariant probability.

PROOF. Let μ be any invariant probability for T. Write $d\sigma_1 = p d\mu$ and $d\sigma_2 = q d\mu$. We compute the Fourier coefficients of μ and use the T invariance of μ to write

$$\hat{\sigma}_1(n) + \hat{\sigma}_2(n) = \hat{\sigma}_1(n) + w^n \hat{\sigma}_2(n)$$

where $w = \exp(-2\pi i \alpha)$. It follows at once that all non-zero Fourier coefficients of σ_2 vanish so $q d\mu$ is a multiple of Lebesgue measure. This proves unique ergodicity.

REMARK. The argument has done two things more than advertized. Namely the invariant probability was shown to be continuous with respect to Lebesgue measure and was even identified. (And it is also clear that Lebesgue measure itself is not invariant for random rotations.)

If strong ergodicity (convergent Cesaro averages of the iterates) were known for random rotations it would settle at once the uniqueness problem. In [9] the author showed that a convex combination of commuting strongly ergodic contractions is strongly ergodic. The non-constant multipliers in the present problem destroy commutativity so that result does not apply.

We will close this section with a description of a blind alley thereby hopefully sparing the reader from repeating the same mis-steps.

The operator which is the random combination of two rotations of the circle can be written as the composition of a rotation operator and another operator which is a random combination of a rotation and the identity. Since a theorem above takes care of the bad factor and the good factor is completely understood one may hope to finish with a general theorem about the product of two strongly ergodic transformations. So the last item of this section is negative in nature. No such general theorem exists. In [10] we constructed a strongly ergodic Markov operator (given by a homeomorphism even) whose square is not strongly ergodic.

REFERENCÉS

1. Robert Atalla and Robert Sine, *Random Markov mappings*, Math. Ann. **221** (1976), 195–199.

2. A. Brunel, *New conditions for existence of invariant measures*, in Lecture Notes **160**, Springer Verlag, 1970.

3. Shaul Foguel, *The Ergodic Theory of Markov Processes*, Van Nostrand, 1969.

4. Shaul Foguel, *More on the Ergodic Theory of Markov Processes*, University of British Columbia Mimeographed Notes, 1973.

5. Benton Jamison and Robert Sine, *Unique ergodicity for certain random translations*, to appear.

6. M. Kac, *A note on learning signal detection*, IRE Trans. Information Theory **8** (1962), 126–128.

7. M. Frank Norman, *Markov Processes and Learning Models*, Academic Press, 1972.

8. M. Frank Norman, Markovian learning processes, SIAM Rev. **16** (1974), 143–162.

9. Robert Sine, *Convex combinations of uniformly mean stable Markov operators*, Proc. Amer. Math. Soc. **51** (1975), 123–126.

10. Robert Sine, *A note on the ergodic properties of homeomorphisms*, Proc. Amer. Math. Soc. **55** (1976), 169–172.

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